Homework 5 solutions

Section 6.5

2. $y' = y^2$

a) $y = \frac{-1}{x}$. Then $y' = \frac{1}{x^2} = y^2$ and $y = \frac{-1}{x}$ satisfies $y' = y^2$.

b) $y = -\frac{1}{x+3}$. Then $y' = \frac{+1}{(x+3)^2} = y^2$ and $y = \frac{-1}{x+3}$ satisfies $y' = y^2$.

c) If $y = -\frac{1}{x+c}$, then $y' = \frac{1}{(x+c)^2} = y^2$, so $y = \frac{-1}{x+c}$ satisfies $y' = y^2$.

4. Show that $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4}\, dt$ satisfies $y' + \frac{2x^3}{1+x^4} y = 1$.

If $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4}\, dt$, then $y' = \frac{1}{\sqrt{1+x^4}} \cdot \sqrt{1+x^4} +$

$$+ \; -\frac{1}{2}(1+x^4)^{-3/2} \cdot 4x^3 \int_1^x \sqrt{1+t^4}\, dt$$

So $y' = 1 + \frac{-2x^3}{(1+x^4)^{3/2}} \int_1^x \sqrt{1+t^4}\, dt$.

Then
$$y' + \frac{2x^3}{1+x^4} y = 1 - \frac{2x^3}{(1+x^4)^{3/2}} \int_1^x \sqrt{1+t^4}\, dt + \frac{2x^3}{1+x^4} \cdot \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4}\, dt$$
$$= 1.$$

So $y = \frac{1}{\sqrt{1+x^4}} \int_1^x \sqrt{1+t^4}\, dt$ satisfies $y' + \frac{2x^3}{1+x^4} y = 1$.

6. $y' = e^{-x^2} - 2xy$, $y(2) = 0$.

Candidate soln: $y = (x-2)e^{-x^2}$.

If $y = (x-2)e^{-x^2}$, $y(2) = 0$ ✓

If $y = (x-2)e^{-x^2}$, $y' = e^{-x^2} + (x-2)e^{-x^2}(-2x)$
$$= e^{-x^2} - 2xy \quad ✓$$

So $y = (x-2)e^{-x^2}$ satisfies $y' = e^{-x^2} - 2xy$, $y(2) = 0$.

10. Solve $\frac{dy}{dx} = x^2\sqrt{y}$, $y > 0$.

Separate variables:

$$\frac{dy}{\sqrt{y}} = x^2 dx$$

Integrate: $\int \frac{dy}{\sqrt{y}} = \int x^2 dx$, so $2\sqrt{y} = \frac{x^3}{3} + C$.

So $\boxed{\sqrt{y} = \frac{x^3}{6} + C}$

12. $\frac{dy}{dx} = 3x^2 e^{-y}$.

Separate variables: $e^y dy = 3x^2 dx$

Integrate: $\int e^y dy = \int 3x^2 dx$

$$\boxed{e^y = x^3 + C}.$$

16. $(\sec x) \frac{dy}{dx} = e^{y + \sin x}$

separate variables: $e^{-y} dy = \cos x \, e^{\sin x} dx$

Integrate: $\int e^{-y} dy = \int \cos x \, e^{\sin x} dx$

$$-e^{-y} = e^{\sin x} + c$$

$$\boxed{e^{-y} = c - e^{\sin x}}$$

Section 15.1

1. $y' = x + y \to$ (d) [Note that if $x = -y$, $y' = 0$, and if x and y increase, y' increases]
2. $y' = y + 1 \to$ (c) [Note that if y is constant, y' is constant, e.g. $y = 1 \to y' = 0$].
3. $y' = \frac{x}{y} \to$ (a) [Note that if $x = 0$, $y' = 0$, if $y = 0$, $y' = \infty$]
4. $y' = y^2 - x^2 \to$ (b) [Note that if $x = y$, $y' = 0$ and if $x = -y$, $y' = 0$].

7. $y = -1 + \int_1^x (t - y(t))\, dt$

$$\boxed{\frac{dy}{dx} = x - y, \quad y(1) = -1}.$$

8. $y = \int_1^x \frac{1}{t}\, dt$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}, \quad y(1) = 0}$$